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STABILITY OF BARS OF VISCO-ELASTIC UNSTABLE MATERIAL UNDER RANDOM PERTURBATIONS†

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The effect of instability of the properties of a material that change deterministically or randomly with time when acted upon by random longitudinal forces is investigated. This investigation is an extension of the results of previous work [1–4] on the stability of visco-stsatic bars with stable properties. Attention is concentrated on ageing, the random distribution of visco-elastic properties, and external damping.

1. STABILITY OF A BAR OF AGEING MATERIAL UNDER THE ACTION OF A RANDOM STATIONARY LONGITUDINAL FORCE

THE MOTION of a bar under the action of a longitudinal force F is described by the equation

$$E(t)Iw^{\mathrm{IV}}+[1+E(t)\mathbf{K}]P=0 \tag{1.1}$$

$$P=Fw''+m\ddot{w}+k\dot{w}$$

$$\mathbf{K}P=\int_{t_0}^{t}K(t,\tau)P(\tau)d\tau,\quad K(t,\tau)=-\frac{\partial}{\partial\tau}\left[\frac{1}{E(\tau)}+C(t,\tau)\right]$$

where k is a damping factor which takes account of external resistance to the motion of the bar and $C(T,\tau)$ is a measure of the creep of the material. The remaining notation is the same as that generally used.

Henceforth we shall assume [5] that

$$C(t,\tau)=\varphi(\tau)[1-e^{-\gamma(t-\tau)}] \tag{1.2}$$

The solution of Eq. (1.1) should of course satisfy the initial and boundary conditions.

We will introduce the functions

$$z_1=\mathbf{K}P$$

$$z_2=\int_{t_0}^{t}\left[\gamma\varphi(\tau)+\frac{\partial\varphi(\tau)}{\partial\tau}\right]e^{-\gamma(t-\tau)}P(\tau)\,d\tau$$

which are solutions of the equations

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$$\dot{z}_1 = \left[\frac{\dot{E}(t)}{E^2(t)} + \gamma\varphi(t)\right]P - \gamma z_2$$

$$\dot{z}_2 = [\dot{\varphi}(t) + \gamma\varphi(t)]P - \gamma z_2 \qquad (1.3)$$

We will assume that the deflection of the bar and its rate of change at the initial instant of time is given by the expressions

$$w(0,x) = w^0 \sin\frac{\pi}{l}x, \quad \dot{w}(0,x) = \upsilon^0 \sin\frac{\pi}{l}x$$

We will seek a solution of Eqs (1.1) and (1.3) in the form

$$w(t,x) = f(t)\sin\frac{\pi}{l}x$$

$$z_1(t,x) = f_3(t)\sin\frac{\pi}{l}x$$

$$z_2(t,x) = f_4(t)\sin\frac{\pi}{l}x$$

After substituting these expressions into Eqs (1.1) and (1.3) we obtain

$$\ddot{f} + 2\varepsilon\dot{f} + \omega^2\left[\frac{E(t)}{E_0} - \alpha(t)\right]f + \frac{E(t)}{m}f_3 = 0$$

$$\dot{f}_3 = m\left[\frac{\dot{E}(t)}{E^2(t)} + \gamma\varphi(t)\right][\ddot{f} + 2\varepsilon\dot{f} - \omega^2\alpha(t)f] - \gamma f_4 \qquad (1.4)$$

$$\dot{f}_4 = m[\dot{\varphi}(t) + \gamma\varphi(t)][\ddot{f} + 2\varepsilon\dot{f} - \omega^2\alpha(t)f] - \gamma f_4$$

Here

$$2\varepsilon = \frac{k}{m}, \quad \omega^2 = \frac{\pi^4 E_0 I}{ml^4}, \quad \alpha(t) = \frac{F(t)l^2}{\pi^2 E_0 I}$$

and E_0 is some (constant) value of the modulus of elasticity (for example, $E_0 = \lim_{t\to\infty} E(t)$, if such a limit exists).

We will represent the system of equations (1.4) in the form of a first-order system $(f_1 = f)$

$$\dot{f}_1 = f_2$$

$$\dot{f}_2 = -2\varepsilon f_2 - \omega^2\left[\frac{E(t)}{E_0} - \alpha(t)\right]f_1 - \frac{E(t)}{m}f_3$$

$$\dot{f}_3 = -\left[\frac{\dot{E}(t)}{E^2(t)} + \gamma\varphi(t)\right]\left[m\omega^2\frac{E(t)}{E_0}f_1 + E(t)f_3\right] - \gamma f_4$$

$$\dot{f}_4 = -[\dot{\varphi}(t) + \gamma\varphi(t)]\left[m\omega^2\frac{E(t)}{E_0}f_1 + E(t)f_3\right] - \gamma f_4$$

In the special case when $E(t) = E_0$, $\varphi(t) = K/E_0 = \text{const}$, it can be seen from (1.2) that $z_3 = z_4$. It then follows from Eqs (1.4) that

$$\dot{f}_1 = f_2$$
$$\dot{f}_2 = -2\varepsilon f_2 - \omega^2[1-\alpha(t)]f_1 - \frac{E_0}{m} f_3$$
$$\dot{f}_3 = -\gamma\left[\frac{\pi^4 I}{l^4}\cdot K f_1 + (1+K) f_3\right]$$

We will further assume that the longitudinal force is a random stationary process $\alpha(t) = \alpha_0 + \alpha_1(t)$ with mathematical expectation $\langle\alpha(t)\rangle = \alpha_0 = \text{const}$ and random fluctuations $\alpha_1(t)$ proportional to "white noise" $\xi(\tau)$, i.e. $\alpha_1(t) = \beta\xi(t)$ and $\beta = \text{const}$. We shall obtain equations for the statistical moments of the functions f_i $(i = 1, 2, 3, 4)$.

The mathematical expectations of these functions are found from the equations [6, 7]

$$\langle \dot{f}_1\rangle = \langle f_2\rangle$$
$$\langle \dot{f}_2\rangle = -2\varepsilon\langle f_2\rangle - \omega^2\left[\frac{E(t)}{E_0} - \alpha_0\right]\langle f_1\rangle - \frac{E(t)}{m}\langle f_3\rangle$$
$$\langle \dot{f}_3\rangle = -\left[\frac{\dot{E}(t)}{E^2(t)} + \gamma\varphi(t)\right]\left[m\omega^2\frac{E(t)}{E_0}\langle f_1\rangle + E(t)\langle f_3\rangle\right] - \gamma\langle f_4\rangle$$
$$\langle \dot{f}_4\rangle = -[\dot{\varphi}(t) + \gamma\varphi(t)]\left[m\omega^2\frac{E(t)}{E_0}\langle f_1\rangle + E(t)\langle f_3\rangle\right] - \gamma\langle f_4\rangle$$

where the angle brackets denote averaging over the set of samples.

If the functions $E(t)$ and $\varphi(t)$ increase with time towards constant values E_0 and φ_0, then one can verify that the bar will be stable in terms of the mathematical expectation of the deflection if the condition

$$\alpha_0 < (1 + E_0\varphi_0)^{-1}$$

is satisfied.

Note that this relation is identical with the condition for the stability of a visco-elastic bar of ageing material in the deterministic version of the problem.

The equations for the second-order moments can be written as follows:

$$\frac{d}{dt}\langle f_1^2\rangle = 2\langle f_1 f_2\rangle$$
$$\frac{d}{dt}\langle f_1 f_2\rangle = \langle f_2^2\rangle - 2\varepsilon\langle f_1 f_2\rangle - \omega^2\left[\frac{E(t)}{E_0} - \alpha_0\right]\langle f_1^2\rangle - \frac{E(t)}{m}\langle f_1 f_3\rangle$$
$$\frac{d}{dt}\langle f_1 f_3\rangle = -\left[\frac{\dot{E}(t)}{E^2(t)} + \gamma\varphi(t)\right]\left[m\omega^2\frac{E(t)}{E_0}\langle f_1^2\rangle + \right.$$
$$\left. + E(t)\langle f_1 f_3\rangle\right] - \gamma\langle f_1 f_4\rangle + \langle f_2 f_3\rangle$$
$$\frac{d}{dt}\langle f_1 f_4\rangle = -[\dot{\varphi}(t) + \gamma\varphi(t)]\left[m\omega^2\frac{E(t)}{E_0}\langle f_1^2\rangle + \right.$$
$$\left. + E(t)\langle f_1 f_3\rangle\right] - \gamma\langle f_1 f_4\rangle + \langle f_2 f_4\rangle$$
$$\frac{d}{dt}\langle f_2^2\rangle = 2\left\{-2\varepsilon\langle f_2^2\rangle - \omega^2\left[\frac{E(t)}{E_0} - \alpha_0\right]\langle f_1 f_2\rangle - \right.$$

$$-\frac{E(t)}{m}\langle f_2 f_3\rangle\} + \omega^4\beta^2\langle f_1^2\rangle$$

$$\frac{d}{dt}\langle f_2 f_3\rangle = -2\varepsilon\langle f_2 f_3\rangle - \omega^2\left[\frac{E(t)}{E_0} - \alpha_0\right]\langle f_1 f_3\rangle -$$

$$-\frac{E(t)}{m}\langle f_3^2\rangle - \left[\frac{\dot{E}(t)}{E^2(t)} + \gamma\varphi(t)\right]\left[m\omega^2\frac{E(t)}{E_0}\langle f_1 f_2\rangle + \right.$$

$$\left. + E(t)\langle f_2 f_3\rangle\right] - \gamma\langle f_2 f_4\rangle$$

$$\frac{d}{dt}\langle f_2 f_4\rangle = -2\varepsilon\langle f_2 f_4\rangle - \omega^2\left[\frac{E(t)}{E_0} - \alpha_0\right]\langle f_1 f_4\rangle -$$

$$-\frac{E(t)}{m}\langle f_1 f_3\rangle - [\dot{\varphi}(t) + \gamma\varphi(t)]\left[m\omega^2\frac{E(t)}{E_0}\langle f_1 f_2\rangle + E(t)\langle f_2 f_3\rangle\right] - \gamma\langle f_2 f_4\rangle$$

$$\frac{d}{dt}\langle f_3^2\rangle = -2\left[\frac{\dot{E}(t)}{E^2(t)} + \gamma\varphi(t)\right]\left[m\omega^2\frac{E(t)}{E_0}\langle f_1 f_3\rangle + E(t)\langle f_3^2\rangle\right] - 2\gamma\langle f_3 f_4\rangle$$

$$\frac{d}{dt}\langle f_3 f_4\rangle = -\left[\frac{\dot{E}(t)}{E^2(t)} + \gamma\varphi(t)\right]\left[m\omega^2\frac{E(t)}{E_0}\langle f_1 f_4\rangle + \right.$$

$$\left. + E(t)\langle f_3 f_4\rangle\right] - \gamma\langle f_4^2\rangle - [\dot{\varphi}(t) + \gamma\varphi(t)]\times$$

$$\times\left[m\omega^2\frac{E(t)}{E_0}\langle f_1 f_3\rangle + E(\dot{t})\langle f_3^2\rangle\right] - \gamma\langle f_3 f_4\rangle$$

$$\frac{d}{dt}\langle f_4^2\rangle = -2[\dot{\varphi}(t) + \gamma\varphi(t)]\left[m\omega^2\frac{E(t)}{E_0}\langle f_1 f_4\rangle + E(t)\langle f_3 f_4\rangle\right] - 2\gamma\langle f_4^2\rangle$$

If the functions $E(t)$ and $\varphi(t)$ tend to constant values as $t \to \infty$, then the root-mean-square stability condition for the bar will be identical with the condition for the solution of the limiting system of Eqs (1.5), obtained from (1.5) as $t \to \infty$ [4, 8], to be stable.

We remark that the nature of the stability of a bar of ageing material, both for its expectation and its mean-square, differs from the nature of the bar stability when the material does not age. In the latter case the stability is asymptotic, which does not happen in the first case.

We considered above a bar for which the material creep measure was governed by expression (1.2). There is a more general expression [5] of the form

$$C(t,\tau) = \varphi(\tau)\sum_{k=0}^{\infty} B_k \exp[-\gamma_k(t-\tau)],\quad B_k, \gamma_k - \text{const}$$

It can be shown that in this case too the solution of the problem, by extending the phase space, can be reduced to considering a system of first-order equations, but of higher order [4].

2. STABILITY OF A VISCO-ELASTIC BAR WHOSE VISCO-ELASTIC CHARACTERISTICS ARE RANDOM FUNCTIONS

We will write the bar equations of motion in the form

$$EI(1-\mathbf{R})w^{\mathrm{IV}} + Fw'' + m\ddot{w} + k\dot{w} = 0 \tag{2.1}$$

$$\mathbf{R}w = \int_0^t R(t,\tau)\, w(\tau, x)\, d\tau$$

where $R(t, \tau)$ is the relaxation kernel of the material, which we will take in the form

$$R(t,\tau) = M(\tau)\exp\left[-\int_\tau^t \kappa(y)\, dy\right]$$

We again look for a solution of Eq. (2.1) in the form of a single half-wave sinusoid and write the equation for its amplitude f

$$\ddot{f} + 2\varepsilon\dot{f} + \omega^2[(1-\mathbf{R}) - \alpha] f = 0 \tag{2.2}$$

We set

$$f_3 = \int_0^t M(\tau)\exp\left[-\int_\tau^t \kappa(y)\, dy\right] f(\tau)\, d\tau \tag{2.3}$$

Then Eqs (2.2) and (2.3) can be represented as a system of first-order differential equations

$$\dot{f}_1 = f_2, \quad \dot{f}_2 = -2\varepsilon f_2 - \omega^2[(1-\alpha) f_1 - \alpha f_3], \quad \dot{f}_3 = M f_1 - \kappa f_3 \tag{2.4}$$

Henceforth we shall assume that α, ε, M and κ are stationary processes with constant mathematical expectations and that the random fluctuations are proportional to white noise

$$\alpha = \alpha_0 + \beta\xi_1, \quad \varepsilon = \varepsilon_0 + \varepsilon_1\xi_2$$
$$M = M_0 + M_1\xi_3, \quad \kappa = \kappa_0 + \kappa_1\xi_4$$
$$\alpha_0 = \langle\alpha\rangle, \quad \varepsilon_0 = \langle\varepsilon\rangle, \quad M_0 = \langle M\rangle, \quad \kappa_0 = \langle\kappa\rangle$$

where ξ_i $(i = 1, \ldots, 4)$ are uncorrelated white noise.

We write system (2.4) as an Ito [6] system of equations

$$df_1 = f_2 dt, \quad df_2 = \{-2\varepsilon_0 f_2 - \omega^2[(1-\alpha_0) f_1 - f_3]\} dt - 2\varepsilon_1 f_2 d\xi_2 + \omega^2 \beta f_1 d\xi_1$$

$$df_3 = (M_0 f_1 - \kappa_0 f_3) dt + M_1 f_1 d\xi_3 - \kappa_1 f_3 d\xi_4 \tag{2.5}$$

If the ξ_i are taken to be white noise in the Stratonovich sense [6], then in the last two equations of system (2.5) one must replace ε_0 by $\varepsilon_0 - \varepsilon_1^2$ and κ_0 by $\kappa_0 - \kappa_1^2/2$.

Equations for the first- and second-order statistical moments [6, 7] follow from (2.5).

Using the Routh–Hurwitz criterion one can show that the motion of the bar is stable with respect to first-order moments if the conditions

$$\alpha_0 < 1 - M_0/\kappa_0 \quad (\xi_4 \text{ — Ito white noise})$$

$$\alpha_0 < 1 - M_0[\kappa_0(1-\rho_0)]^{-1} \quad (\xi_4 \text{ — Stratonovich white noise})$$

$$\rho = \kappa_1^2/(2\kappa_0)$$

are satisfied.

The first of the conditions is identical with the bar stability condition in the deterministic problem. The second condition is more restrictive (for $\kappa_1^2/2 < \kappa_0$). According to the second inequality, beyond some value of κ_1 the bar is only stable when the mean longitudinal force is a strain.

The conditions for the motion of the bar to be stable with respect to the second-order moments are very complex. For Ito white noise we obtain

$$\begin{aligned}&-[\omega^2\beta^2 M_0 - 2(\varepsilon_0 - \varepsilon_1^2)M_1^2 - 4(1-\alpha_0)(\varepsilon_0 - \varepsilon_1^2)M_0]+\\&+(1-\alpha_0)(2\kappa_0 - \kappa_1^2)[\omega^2\beta^2/2 + 2(1-\alpha_0)(\varepsilon_0 - \varepsilon_1^2) - M_0]+\\&+(2M_0^2 + M_1^2\kappa_0) + (2\varepsilon_0 + \kappa_0)(2\kappa_0 - \kappa_1^2)\times\\&\times[-2(1-\alpha_0)\kappa_0(\varepsilon_0 - \varepsilon_1^2)/\omega^2 + \beta^2\kappa_0/2 + 2M_0(\varepsilon_0 - \varepsilon_1^2)/\omega^2] < 0\end{aligned} \tag{2.6}$$

Analysis of this reaction for the general case is very difficult. We will therefore consider some special cases that are of practical interest.

1. *The external resistance and visco-elastic material properties are deterministic* ($\varepsilon_1 = M_1 = \kappa_1 = 0$). The stability condition that follows from inequality (2.6) are identical with that obtained previously [4]. The restriction on β^2 turns out to be independent of the particular white noise ξ_1 that we are considering (whether Ito or Stratonovich).

2. *The external resistance and statistical spread of* M *are ignored* ($\varepsilon_0 = \varepsilon_1 = M_1 = 0$). It follows from (2.6) that

$$\omega^2\beta^2 < 2M_0\left\{1 + \frac{\kappa_0^2}{\omega^2}\left[1 - \alpha_0 - \frac{M_0}{\kappa_0}\left(1 - \frac{\kappa_1^2}{2\kappa_0}\right)^{-1}\right]^{-1}\right\}^{-1} \tag{2.7}$$

One can verify that when $\rho < 1$ in the case under consideration, stronger restrictions are imposed on the scatter of the longitudinal force than in the preceding case (with $\varepsilon_0 = 0$).

If ξ_3 is taken to be Stratonovich white noise, then in inequality (2.7) one should replace κ_0 by the difference $\kappa_0 - \kappa_1^2/2$.

3. *The external friction and statistical spread of* κ *are ignored* ($\varepsilon_0 = \varepsilon_1 = \kappa_1 = 0$). Relation (2.6) acquires the form

$$\begin{aligned}&\omega^2\beta^2 < 2M_0[\delta - M_1^2/(2M_0)](\delta + \kappa_0^2/\omega^2)^{-1}\\&\delta = 1 - \alpha_0 - M_0/\kappa_0\end{aligned} \tag{2.8}$$

It is clear that in this case the upper restriction on β^2 is stiffer than in case 1 (with $\varepsilon_0 = 0$). Condition (2.8) is independent of whether the white noise ξ_3 is that of Ito or Stratonovich.

4. *Deterministic visco-elastic material characteristics* ($M_1 = \kappa_1 = 0$). From inequality (2.6) we have

$$\begin{aligned}&\omega^2\beta^2 < 2\delta[\delta + \kappa_0(2\varepsilon_0 + \kappa_0)/\omega^2]^{-1}\times\\&\times[2(1-\alpha_0)(\varepsilon_0 - \varepsilon_1^2) + M_0 + 2\kappa_0(2\varepsilon_0 + \kappa_0)(\varepsilon_0 - \varepsilon_1^2)/\omega^2]\end{aligned}$$

and we again conclude that mean-square stability of the bar is possible at smaller values of the intensity coefficient β than in case 1.

If ξ_2 is Stratonovich white noise, then ε_0 must be replaced by $\varepsilon_0 - \varepsilon_1^2$.

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